Episode 6

Conservation Laws for Particles
Work-Power- Kinetic Energy relations for a single particle

ENGN0040: Dynamics and Vibrations

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Topics for todays class

Work, Power, Kinetic Energy relations for a single particle

1. Definition of rate of work (power) developed by a force
2. Definition of total work done by a force
3. Definition of kinetic energy of a particle
4. Power-work-kinetic energy relations for a single particle
5. Applications

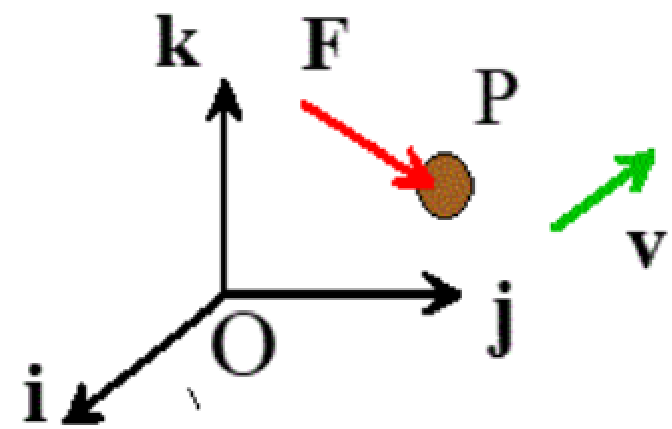
4) Conservation Laws for particles

4.1 Power-Work-Kinetic energy relations for a single particle

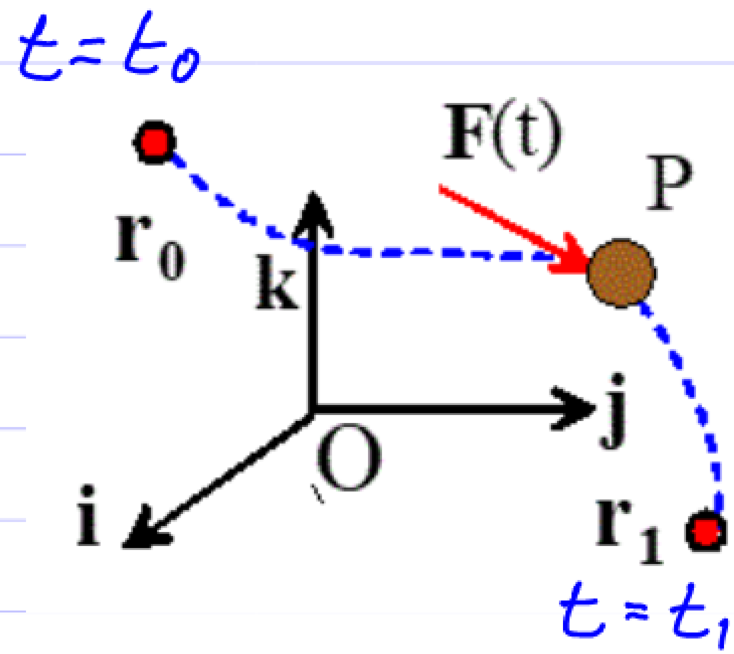
4.1.1 Definitions of power and work

Power of a force

$P = \underline{F} \cdot \underline{v}$ Units: "Watts"
$kg\, m^2\, s^{-3}$



Work done by a force
(over time interval $t_0 < t < t_1$



$$W = \int_{t_0}^{t_1} P\, dt$$

$$= \int_{t_0}^{t_1} \underline{F} \cdot \underbrace{\underline{v}\, dt}_{d\underline{r}}$$

$$= \int_{\underline{r}_0}^{\underline{r}_1} \underline{F} \cdot d\underline{r}$$

Units "Joules"
$kg\ m^2\ s^{-2}$

Inverse relation: Given $W(x, y, z)$

$$\underline{F} = \nabla W = \frac{\partial W}{\partial x}\underline{i} + \frac{\partial W}{\partial y}\underline{j} + \frac{\partial W}{\partial z}\underline{k}$$

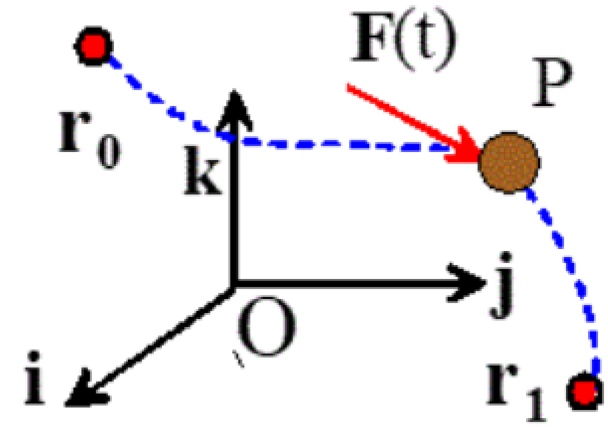
4.1.2 Definition of Kinetic Energy

$$T = \frac{1}{2} m |\underline{v}|^2 = \frac{1}{2} m \underline{v} \cdot \underline{v} = \frac{1}{2} m V^2$$

Units: also Joules

4·1·3 Power-Work-Kinetic Energy relations for a single particle

Let $\underline{F}$ be total force on particle



$$P = \underline{F} \cdot \underline{V} \qquad W = \int_{t_0}^{t_1} P \, dt$$

$T = \frac{1}{2} m V^2$ Let $T = T_0$ @ time $t = t_0$

$T = T_1$ @ time $t = t_1$

① $$\boxed{P = \frac{dT}{dt}}$$ (Power-KE relation)

② $$\boxed{W = T_1 - T_0}$$ (Work-KE relation)

Proof $\quad \underline{F} = m\underline{a} = m \dfrac{d\underline{v}}{dt}$

$$\Rightarrow \quad \underline{F} \cdot \underline{v} = m \frac{d\underline{v}}{dt} \cdot \underline{v}$$

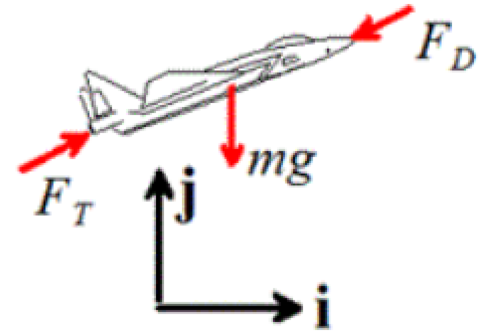
Note $\quad \dfrac{d}{dt}\left\{ \frac{1}{2} m\, \underline{v} \cdot \underline{v} \right\} = \underbrace{\frac{1}{2} m \left\{ \frac{d\underline{v}}{dt} \cdot \underline{v} + \underline{v} \cdot \frac{d\underline{v}}{dt} \right\}}_{m \frac{d\underline{v}}{dt} \cdot \underline{v}}$

Hence $\quad P = \dfrac{dT}{dt}$

Separate variables & integrate

$$\int_{t_0}^{t_1} P\, dt = \int_{T_0}^{T_1} dT \Rightarrow W = T_1 - T_0$$

4.1.4: Example: Calculating power of a force. An aircraft with mass 15000 kg flying at 200 knots (102 m/s) climbs at 1000 ft/min (5.1 m/s). Calculate the rate of work done on the aircraft by gravity

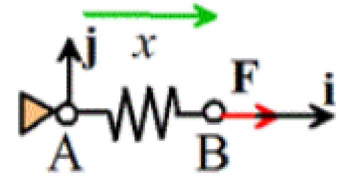


Formula: $P = \underline{F} \cdot \underline{v} = -mg\underline{j} \cdot (v_x \underline{i} + v_y \underline{j})$

$= -mg\, v_y$

$= \boxed{-750 \text{ kW}}$

4.1.5: Example: An external force *F(x)* stretches a spring with stiffness *k* an un-stretched length L_0 from an initial length L_1 to a new length L_2 . Calculate the work done by the force.



Formula : $W = \int_{\underline{r}_0}^{\underline{r}_1} \underline{F} \cdot d\underline{r}$ $\qquad \underline{F} = k(x - L_0)\,\underline{i}$

$d\underline{r} = dx\,\underline{i}$

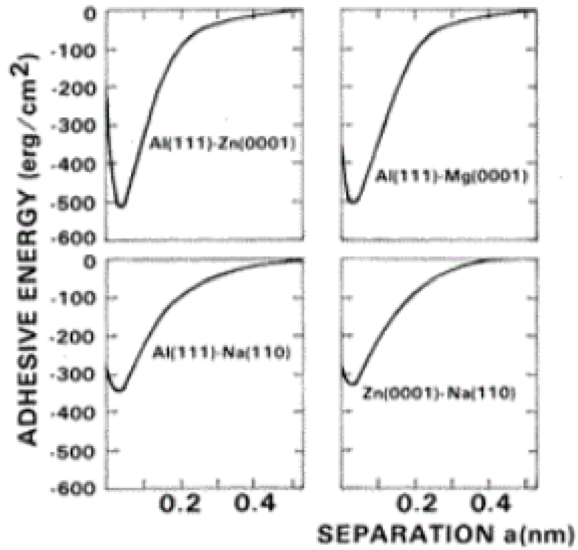
$$\Rightarrow W = \int_{L_1 \underline{i}}^{L_2 \underline{i}} k(x - L_0)\,\underline{i} \cdot dx\,\underline{i}$$

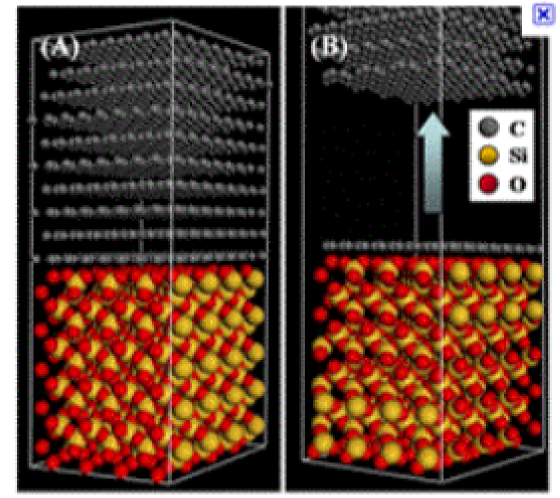
$$W = \int_{L_1}^{L_2} k(x - L_0)\,dx$$

$$\Rightarrow \boxed{W = \frac{1}{2}k(L_2 - L_0)^2 - \frac{1}{2}k(L_1 - L_0)^2}$$

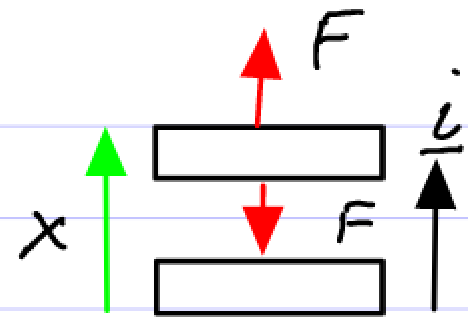
4.1.6: Example: The work per unit area required to separate the interface between two crystals by a distance *x* can be approximated by the 'Universal Binding Energy Relation'

$$W(x) = -E_0\left(1+\frac{x}{d}\right)\exp\left(-\frac{x}{d}\right)$$





1. Calculate the force of attraction (per unit area) between the planes as a function of x
2. Calculate the stiffness of the bond between the planes



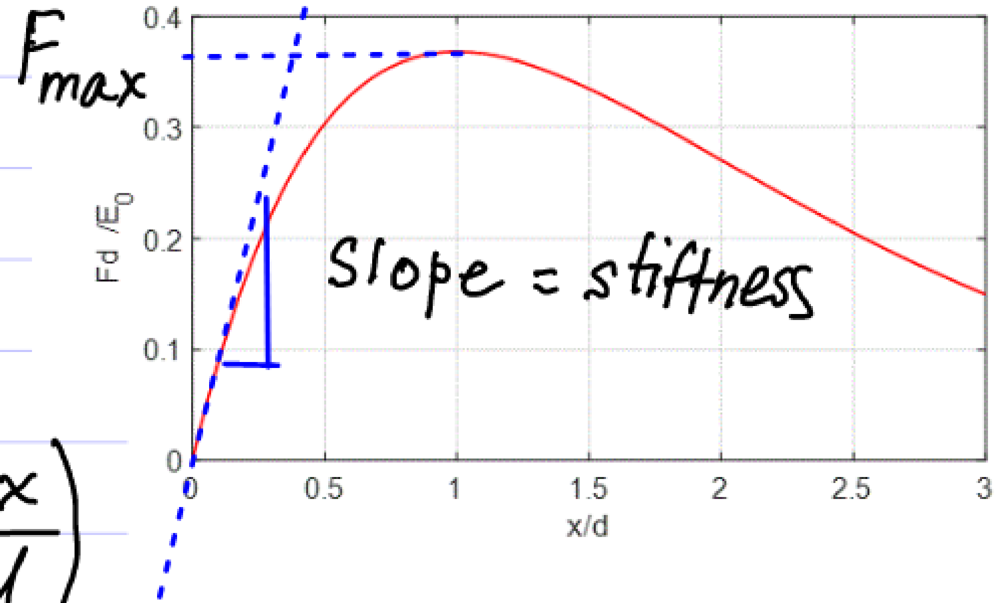
Formula: $\underline{F} = \nabla W = \frac{\partial W}{\partial x}\underline{i} + \frac{\partial W}{\partial y}\underline{j} + \frac{\partial W}{\partial z}\underline{k}$

$$\Rightarrow \underline{F} = \left\{ \frac{-E_0}{d}\exp\left(\frac{-x}{d}\right) - E_0\left(1+\frac{x}{d}\right)\left(\frac{-1}{d}\right)\exp\left(\frac{-x}{d}\right)\right\}\underline{i}$$

$$\Rightarrow \boxed{\underline{F} = \frac{E_0 x}{d^2}\exp\left(-\frac{x}{d}\right)\underline{i}}$$

By definition stiffness

$$k = \left.\frac{\partial F}{\partial x}\right|_{F=0}$$



Here $\dfrac{\partial F}{\partial x} = \dfrac{E_0}{d^2} \exp\left(\dfrac{-x}{d}\right)\left(1 - \dfrac{x}{d}\right)$

$F = 0$ when $x = 0 \Rightarrow$ $\boxed{k = \dfrac{E_0}{d^2}}$

Also F_{max} (force to break interface) occurs at $\partial F / \partial x = 0 \Rightarrow x = d$

$$\Rightarrow \boxed{F_{max} = \frac{E_0}{d} \exp(-1) = 0.37 \frac{E_0}{d}}$$

4.1.7: Example: Estimate time for Ferrari Tributo to reach 200 km / hr

Specifications

Max engine power 530 kW
Mass 1435 kg
Frontal area 4 m^2
Drag coefficient 0.32
Drag force $F_D = \frac{1}{2}\rho C_D A V^2$
Air density $\rho = 1.2 \ kg/m^3$

Assume average engine power is constant at $2P_{\max}/3$

Approach: use power - KE relation for a particle

$$P_{TOTAL} = dT/dt$$

Engine and air drag do work on vehicle

$$P_{TOTAL} = 2P_{max}/3 + P_{drag}$$

$$P_{drag} = \underline{F}_{drag} \cdot \underline{V} = \underbrace{-\frac{1}{2}\rho C_D A V^2}_{\text{magnitude}} \underbrace{\frac{\underline{V}}{V}}_{\text{Direction}} \cdot \underline{V}$$

Note $\underline{V} \cdot \underline{V} = V^2 \Rightarrow P_{drag} = -\frac{1}{2}\rho C_D A V^3$

Hence $2P_{max}/3 - \frac{1}{2}\rho C_D A V^3 = \frac{d}{dt}\left\{\frac{1}{2} m V^2\right\}$

$$\Rightarrow 2P_{max}/3 - \frac{1}{2}\rho C_D A V^3 = m V \frac{dV}{dt}$$

Separate variables & integrate

$$\int_0^t dt = \int_0^{V_{max}} \frac{m V}{2P_{max}/3 - \frac{1}{2}\rho C_D A V^3} \, dV$$

Evaluate integral with "Live Script"

4.1.8: Example: The Pipistrel Velis Electro is the world's first type certified electric aircraft. It has specifications

Empty weight 430 kg
Max Engine Power 60 kW
Lift/drag ratio 8:1
Endurance 50 mins (plus 30 min reserve)
Cruise speed 167 km/hr

Estimate (with 100kg pilot):

1. The power consumption during level cruise
2. The battery capacity
3. The maximum climb rate at cruise speed

(1) Power-KE formula $P_{total} = dT/dt$

$$P_{total} = P_{engine} + P_{Lift} + P_{drag} + P_{gravity}$$

Level cruise $\Rightarrow$ $P_{Lift} = P_{gravity} = dT/dt = 0$

$\Rightarrow$ $P_{engine} = -P_{drag}$

From 4.1.7 $P_{drag} = -|F_{drag}|\,\bar{V}$ $\quad F_{drag} = \dfrac{F_{Lift}}{8} = \dfrac{mg}{8}$

$$\Rightarrow \boxed{P_{engine} = \frac{mg\bar{V}}{8} = 30\text{ kW}}$$

(2) Battery capacity = $P_{engine} \times$ endurance

$\Rightarrow$ Capacity $= 30\ kW \times \frac{(50+30)}{60}$ hrs

$$\boxed{\text{Capacity} = 45\ \text{kW hrs}}$$

(3) Climb rate

Assume const speed (167 km/hr)

$$\underbrace{P_{engine}}_{50\ kW} + \underbrace{P_{lift}}_{=0} + \underbrace{P_{drag}}_{-30\ kW} + \underbrace{P_{gravity}}_{-mg\,V_y} = 0$$

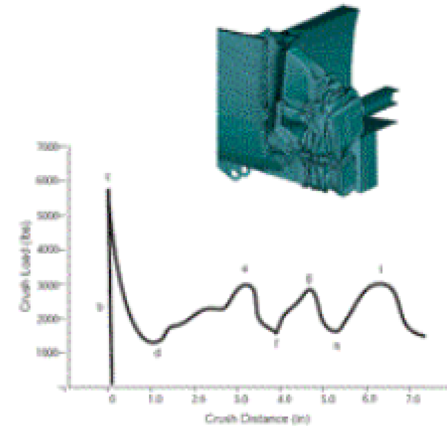
(see 4.1.4)

$$\Rightarrow \boxed{V_y = \frac{20 \times 10^3}{530 \times 10} = 3.8\ m/s}$$

4.1.9: Example: Estimate the length of the crumple zone required to provide protection from 30 mph crash

Assume:

1. Crumple zone exerts a constant force
2. Max acceleration may not exceed 8g

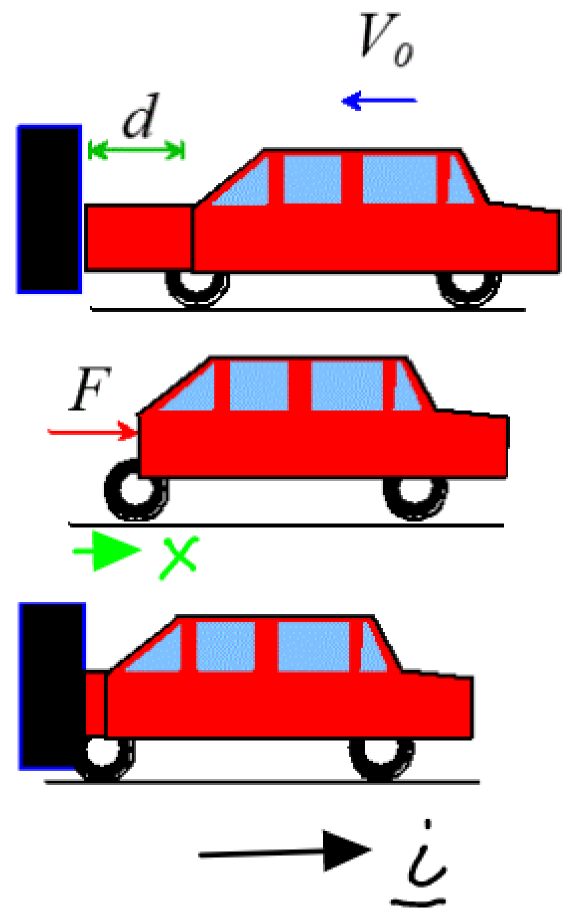


Approach: (1) $\underline{F} = m\underline{a}$ to get F

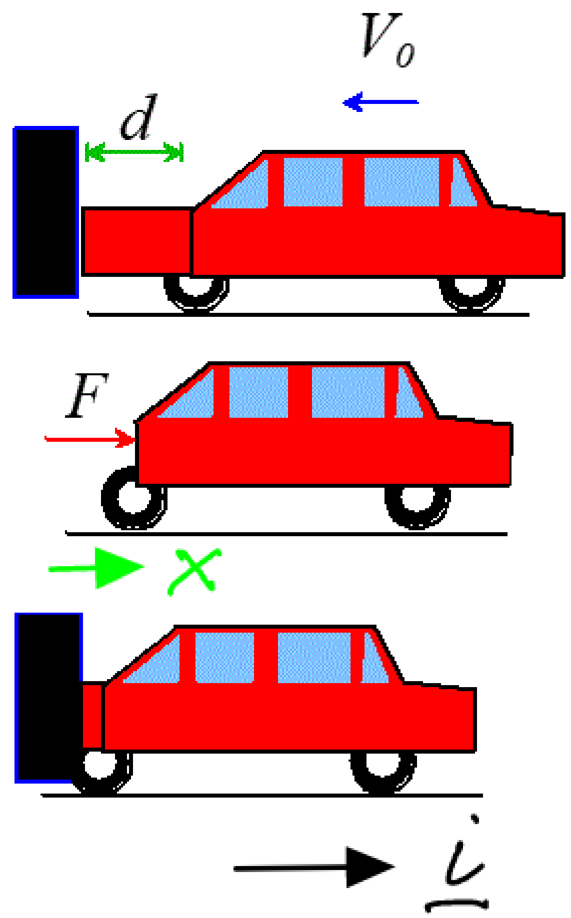
(2) Work-KE to find d

(1) $\underline{F} = m\underline{a} \Rightarrow F\underline{i} = m a_x \underline{i}$

$a_x < 8g \Rightarrow \boxed{F < 8mg}$



Work - KE



$$W = T_1 - T_0$$

$$W = \int_{d\underline{i}}^{0\underline{i}} F\underline{i} \cdot dx\,\underline{i}$$

$$T_1 = 0 \qquad T_0 = \frac{1}{2} m \bar{V}_0^2$$

$$\Rightarrow \int_d^0 F\,dx = -\frac{1}{2} m \bar{V}_0^2$$

$$\Rightarrow -dF = -\frac{1}{2} m \bar{V}_0^2 \Rightarrow d = \frac{1}{2} m \bar{V}_0^2 / F$$

$$F < 8mg \Rightarrow \boxed{d > \frac{V_0^2}{16g}}$$

$d \approx 1.2\ m$
for $\bar{V}_0 = 30\,mph$